BINOMIAL THEOREM

1. STATEMENT OF BINOMIAL THEOREM

$(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \dots\dots\dots + {}^nC_na^n$ (where $n \in N$)

- ${}^nC_0, {}^nC_1, {}^nC_2, \dots\dots, {}^nC_n$ are binomial coefficients ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

General Term $= T_{r+1} = {}^nC_r x^{n-r}a^r$

- There are (n+1) terms in the expansion of $(x + a)^n$.
- The sum of powers of a and x in each term of expansion is n.
- The binomial coefficients in the expansion of $(x + a)^n$ equidistant from the beginning and the end are equal.

2. GREATEST BINOMIAL COFFICIENT

- If n is even : When $r = \dfrac{n}{2}$ i.e. ${}^nC_{n/2}$ takes maximum value.
- If n is odd : $r = \dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ i.e. ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ and take maximum value.

3. MIDDLE TERM OF THE EXPANSION

- If n is even $T_{\left(\frac{n}{2}+1\right)}$ is the middle term. So the middle term $T_{\left(\frac{n}{2}+1\right)} = {}^nC_{n/2}x^{n/2}y^{n/2}$
- If n is odd $T_{\left(\frac{n+1}{2}\right)}$ and $T_{\left(\frac{n+3}{2}\right)}$ are middle terms. So the middle terms are

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\left(\frac{n-1}{2}\right)}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}} \text{ and } T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\left(\frac{n+1}{2}\right)}x^{\frac{n-1}{2}}y^{\frac{n+1}{2}}$$

4. TO DETERMINE A PARTICULAR TERM IN THE EXPANSION

In the expansion of $\left(x^\alpha \pm \dfrac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1}, then r is given by

$$r = \frac{n\alpha - m}{\alpha + \beta}$$

The term which is independent of x, occurs in T_{r+1}, then r is $r = \dfrac{n\alpha}{\alpha + \beta}$

5. BINOMIAL COEFFICIENT PROPERTIES

(1) $C_0 + C_1 + C_2 + \dots\dots\dots + C_n = 2^n$

(2) $C_0 - C_1 + C_2 - C_3 + \dots\dots\dots + (-1)^n C_n = 0$

(3) $C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(4) $C_0C_r + C_1C_{r+1} + \ldots\ldots\ldots + C_{n-r}C_n = {}^{2n}C_{n-r} = \dfrac{2n!}{(n+r)!\,(n-r)!}$

(if r = 0) $C_0^{\ 2} + C_1^{\ 2} + C_2^{\ 2} + \ldots\ldots\ldots + C_n^{\ 2} = \dfrac{2n!}{n!\,n!}$

(if r = 1) $C_0\,C_1 + C_1\,C_2 + C_2\,C_3 + \ldots\ldots\ldots + C_{n-1}\,C_n = {}^{2n}C_{n-1} = \dfrac{2n!}{(n+1)!(n-1)!}$

(5) $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots + nC_n = n.2^{n-1}$

(6) $C_1 - 2C_2 + 3C_3 - \ldots\ldots\ldots + (-1)^n \,.\, nC_n = 0$

(7) $C_0 + 2C_1 + 3C_2 + \ldots\ldots\ldots + (n+1)C_n = 2^{n-1}\ (n+2)$

(8) $C_0 + \dfrac{C_1}{2}x + \dfrac{C_2}{3}x^2 + \dfrac{C_3}{4}x^3 + \ldots\ldots + \dfrac{C_n}{n+1}x^n = \dfrac{(1+x)^{n+1}-1}{(n+1)x}$

$\Rightarrow C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}\quad (x = 1)$

$\Rightarrow C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} + \ldots\ldots\ldots + \dfrac{(-1)^n.C_n}{n+1} = \dfrac{1}{(n+1)}\quad (x = -1)$

(9) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots\ldots + (-1)^n Cx^2 = \left\{\begin{matrix} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\ n_{C_{n/2}} & \text{if } n \text{ is even}\end{matrix}\right\}$

6. NUMERICALLY GREATEST TERM OF BINOMIAL EXPANSION

$(a + x)^n = C_0a^n + C_1a^{n-1}x + \ldots\ldots\ldots\ldots + C_nx^n.$

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\right| \left|\frac{x}{a}\right| = \left|\frac{n-r+1}{r}\right| \left|\frac{x}{a}\right|$$

Take $\left|\dfrac{n-r+1}{r}\right| \left|\dfrac{x}{a}\right| \geq 1$ $\quad\{\text{As } |T_{r+1}| \geq |T_r|\}$

$$r \leq \frac{n+1}{1+\left|\frac{a}{x}\right|}$$

So greatest term will be T_{r+1} where $r = \left[\dfrac{n+1}{1+\left|\frac{a}{x}\right|}\right]$

[.] denotes greatest integer function.

Note : If $\dfrac{n+1}{1+\left|\frac{a}{x}\right|}$ itself is a natural number, then $T_r = T_{r+1}$ and both the terms are numerically greatest.

7. BINOMIAL THEOREM FOR ANY INDEX

If $n \in Q$, $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + \frac{n(n-1)(n-2)\ldots..(n-r+1)}{r!}x^r + \ldots\ldots..\infty$$

Note : In this case there are infinite terms in the expansion.

Some Important Expansions :

If $|x| < 1$ and $n \in Q$ then

(a) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)\ldots..(n+r-1)}{r!}x^r + \ldots..$

(b) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)\ldots..(n+r-1)}{r!}(-x)^r + \ldots..$

By putting n = 1, 2, 3 in the above results, we get the following results-

- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots.$
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots + (-x)^r + \ldots\ldots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.. + (r+1)\, x^r + \ldots..$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots + (r+1)\,(-x)^r + \ldots\ldots$
- $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}x^r + \ldots.$
- $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots\ldots + \frac{(r+1)(r+2)}{2!}(-x)^r + \ldots\ldots..$

8. SOME IMPORTANT RESULTS

(i) If the coefficient of the r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1+x)^n$ are in H.P. then $n + (n-2r)^2 = 0$

(ii) If coefficient of r^{th}, $(r+1)^{th}$, and $(r+2)^{th}$ terms in the expansion of $(1+x)^n$ are in A.P. then $n^2 - n(4r+1) + 4r^2 - 2 = 0$

(iii) Number of terms in the expansion of $(x_1 + x_2 + \ldots\ldots x_r)^n$ is ${}^{(n+r-1)}C_{r-1}$